Modular Arithmetic I: Algebraic Cycles

BMC Int I Fall 2023

September 13, 2023

1 Cycling Remainders

Exercise 1.1. *Define a sequence* a_n *by* $a_1 = 1$ *and* $a_{n+1} = 3^{a_n}$*. What are the last two digits of* a_{100}*?*

Definition 1.2. *We say that* $a \equiv b \pmod{n}$ *if* $n \mid a - b$*, or* n *divides the difference* $a - b$*.*

Exercise 1.3. *Show that* $a_1 \equiv b_1 \pmod{n}$ *and* $a_2 \equiv b_2 \pmod{n}$*, then which of the following are true:*

- $a_1 + a_2 \equiv b_1 + b_2 \pmod{n}$;
- $a_1 a_2 \equiv b_1 b_2 \pmod{n}$;
- $a_1/a_2 \equiv b_1/b_2 \pmod{n}$;
- $a_1^{a_2} \equiv b_1^{b_2} \pmod{n}$.

For the ones that are not always true, are they true if we impose some conditions on a, b, n*?*

Exercise 1.4. *Prove that* $1^n + 2^n + \cdots + (n-1)^n$ *is divisible by* n *for any odd* $n > 1$*.*

Exercise 1.5. *What is the remainder* $2^{2023} \pmod{3}$ *and* $2018^{2020} \pmod{2019}$*? What about* $3^{1995} \pmod{5}$ *and* $4^{2041} \pmod{14}$*?*

Exercise 1.6. *Calculate* $3n \pmod{7}$ *for* $n = 1, 2, 3, 4, 5, 6$*. What do you notice? What is* $3^6 \pmod{7}$*?*

Theorem 1.7. *If* p *is a prime number and* $p \nmid a$*, then* $a^{p-1} \equiv 1 \pmod{p}$*.*

Exercise 1.8. *Let* $\phi(n)$ *denote the number of integers small than* n *that are relatively prime to* n*. What is* $\phi(10)$*? What is* $3^{\phi(10)} \pmod{10}$*?*

Exercise 1.9. *What is* $\phi(27)$*?* $\phi(40)$*?* $\phi(100)$*?*

Exercise 1.10. *Define a sequence* a_n *by* $a_1 = 1$ *and* $a_{n+1} = 3^{a_n}$*. What are the last two digits of* a_{100}*?*

Exercise 1.11. *Does the sequence defined by* $a_1 = k, a_{n+1} = k^{a_n}$ *eventually become constant mod* m*? Show it by following the three steps:*

1. *Show that the sequence* $a_n \pmod{m}$ *is eventually cyclic (it repeats in a cycle)*
2. *Now assume that this is not true for some* m, k*. Let* M *be the smallest such* m *that this is not true. What can we say about the numbers* $b_i \pmod{\phi(M)}$ *defined as* $a_i = k^{b_i}$*.*
3. *Conclude that we get a contradiction and no such* (m, k) *exists.*

Exercise 1.12. *Show that there exists an* m *such that* $\frac{2^m - m}{2023}$ *is an integer.*

Exercise 1.13. *Show that there exists an* m *such that* $\frac{2^m + 3^m + 5^m - m}{235}$ *is an integer.*